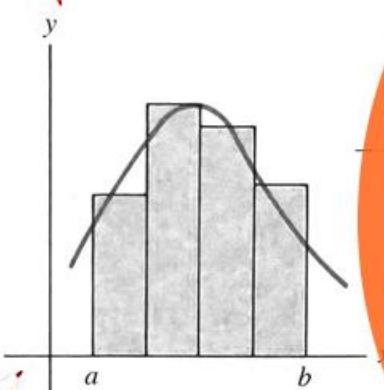
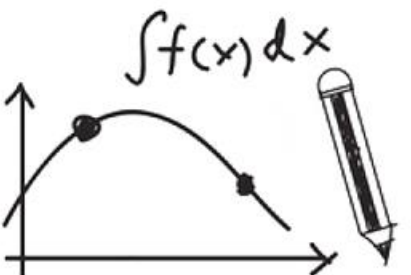


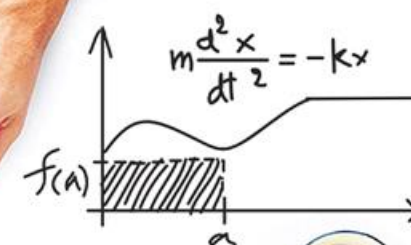
$$x^2 - 3x - 4 = 0$$
$$4x^2 - 3x - 1 = 0$$



Calculus(I)

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$F = mg = ma = m \frac{d^2h}{dt^2}$$



Gottfried Wilhelm Leibniz

$$\frac{dA}{dt} = \frac{dB}{dt} = -\frac{dC}{dt} = \frac{dD}{dt} = (c_1)T^{\frac{1}{2}}AB - (c_2)T^{\frac{1}{2}}CD$$

$$m \frac{d^2x}{dt^2} = -kx - f \frac{dx}{dt} + A \sin(\omega t)$$

$$y' = \text{and } v' = -ky - fv + A \sin(\omega t)$$

$$m \frac{d^2x}{dt^2} = -kx$$

$$x = A e^{dT}$$
$$\frac{dA}{dt} = (c_1)(T - T)$$

$$\frac{df(x)}{dx}$$

$$\frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$L(x+h), f(x+h)$$

Limits Involving Trigonometric Functions

Lecturer: Xue Deng

Problem Introduction



How about $\lim_{t \rightarrow c} \sin t$, $\lim_{t \rightarrow 0} \frac{\sin t}{t}$ and $\lim_{t \rightarrow 0} \frac{1 - \cos t}{t}$?

Theorem A

Limits of Trigonometric Functions

For every real number c in the function's domain

$$1 \quad \lim_{t \rightarrow c} \sin t = \sin c$$

$$2 \quad \lim_{t \rightarrow c} \cos t = \cos c$$

$$3 \quad \lim_{t \rightarrow c} \tan t = \tan c$$

$$4 \quad \lim_{t \rightarrow c} \cot t = \cot c$$

$$5 \quad \lim_{t \rightarrow c} \sec t = \sec c$$

$$6 \quad \lim_{t \rightarrow c} \csc t = \csc c$$

Theorem B Special Trigonometric Limits

$$1 \quad \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$$

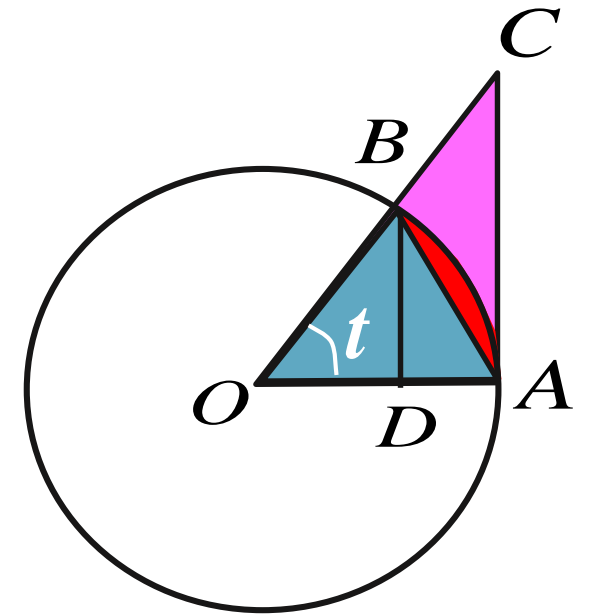
$$2 \quad \lim_{t \rightarrow 0} \frac{1 - \cos t}{t} = 0$$

Proof: 1

Let unit-circle O , $\angle AOB = t$, $(0 < t < \frac{\pi}{2})$

Then $\sin t = BD$, $t = \widehat{AB}$, $\tan t = AC$,

Area of $\triangle AOB < \text{Area of Sector } AOB < \text{Area of } \triangle AOC$



Theorem B Special Trigonometric Limits

namely $\frac{1}{2} \sin t < \frac{1}{2} t < \frac{1}{2} \tan t$

$$\therefore \sin t < t < \tan t, \quad \therefore 1 < \frac{t}{\sin t} < \frac{1}{\cos t},$$

$$\cos t < \frac{\sin t}{t} < 1,$$

when $-\frac{\pi}{2} < t < 0$ this formula still holds.

$$\therefore \lim_{t \rightarrow 0} \cos t = 1, \quad \therefore \lim_{t \rightarrow 0} 1 = 1,$$

$$\therefore \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 \quad \left(\lim_{t \rightarrow 0} \frac{t}{\sin t} = 1 \right)$$

By Squeeze Theorem

Theorem B Special Trigonometric Limits



$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

(1) It is a limit of type $\left[\frac{0}{0} \right]$;

(2) $\sin \square$ and \square , the corresponding \square are same expressions, tend to 0

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 1 \quad \text{X} \quad \text{?} \quad (\because \text{It is not the type of } \left[\frac{0}{0} \right].)$$

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0 \quad \text{✓}$$

Example 1

$$\text{Find } \lim_{t \rightarrow 0} \frac{t^2}{t+1} \cdot \cos t$$

$$\begin{aligned} & \text{✎} = \lim_{t \rightarrow 0} \frac{t^2}{t+1} \cdot \lim_{t \rightarrow 0} \cos t \end{aligned}$$

$$= 0 \cdot 1$$

$$= \mathbf{0}.$$

Example 2

$$(1) \lim_{x \rightarrow 0} \frac{x}{\tan x}$$

$$\begin{array}{l} \text{✎} \\ = \lim_{x \rightarrow 0} \frac{x}{\sin x} \cos x \end{array}$$

$$= \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \lim_{x \rightarrow 0} \cos x$$

$$= \mathbf{1.}$$


$$(2) \lim_{x \rightarrow 0} \frac{\sin^3 \sqrt[3]{x}}{3x}$$

$$\begin{array}{l} \text{✎} \\ = \frac{1}{3} \lim_{x \rightarrow 0} \left(\frac{\sin \sqrt[3]{x}}{\sqrt[3]{x}} \right)^3 \end{array}$$

$$= \frac{\mathbf{1}}{\mathbf{3}}.$$

Example 3


$$(1) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$$


$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2}$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$= \frac{1}{2}.$$

$$(2) \lim_{n \rightarrow \infty} n \sin \frac{2}{n}$$


$$= \lim_{n \rightarrow \infty} 2 \frac{\sin \frac{2}{n}}{\frac{2}{n}}$$

$$= 2.$$

Summary of Theorems

$$1 \quad \lim_{t \rightarrow c} \sin t = \sin c$$

$$3 \quad \lim_{t \rightarrow c} \tan t = \tan c$$

$$5 \quad \lim_{t \rightarrow c} \sec t = \sec c$$

$$7 \quad \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$$

$$2 \quad \lim_{t \rightarrow c} \cos t = \cos c$$

$$4 \quad \lim_{t \rightarrow c} \cot t = \cot c$$

$$6 \quad \lim_{t \rightarrow c} \csc t = \csc c$$

$$8 \quad \lim_{t \rightarrow 0} \frac{1 - \cos t}{t} = 0$$

Questions and Answers

Q1: Find $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$.




$$= \lim_{x \rightarrow 0} 3 \cdot \frac{\sin 3x}{3x}$$

$$= 3 \cdot 1$$

$$= 3.$$

Questions and Answers

Q2: Find $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x}$.


$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \frac{1 - \cos x}{x} \\ &= 1 \cdot 0 \\ &= 0. \end{aligned}$$

$$\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1, \quad \lim_{t \rightarrow 0} \frac{1 - \cos t}{t} = 0$$

Questions and Answers

Q3: Find $\lim_{x \rightarrow 0} \frac{\sin 4x}{\tan x}$.



$$= \lim_{x \rightarrow 0} \frac{\sin 4x}{\sin x / \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{4x}{\sin x} \cdot \lim_{x \rightarrow 0} \cos x$$

$$= 1 \cdot 4 \cdot 1$$

$$= 4.$$

Limits Involving Trigonometric Functions

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