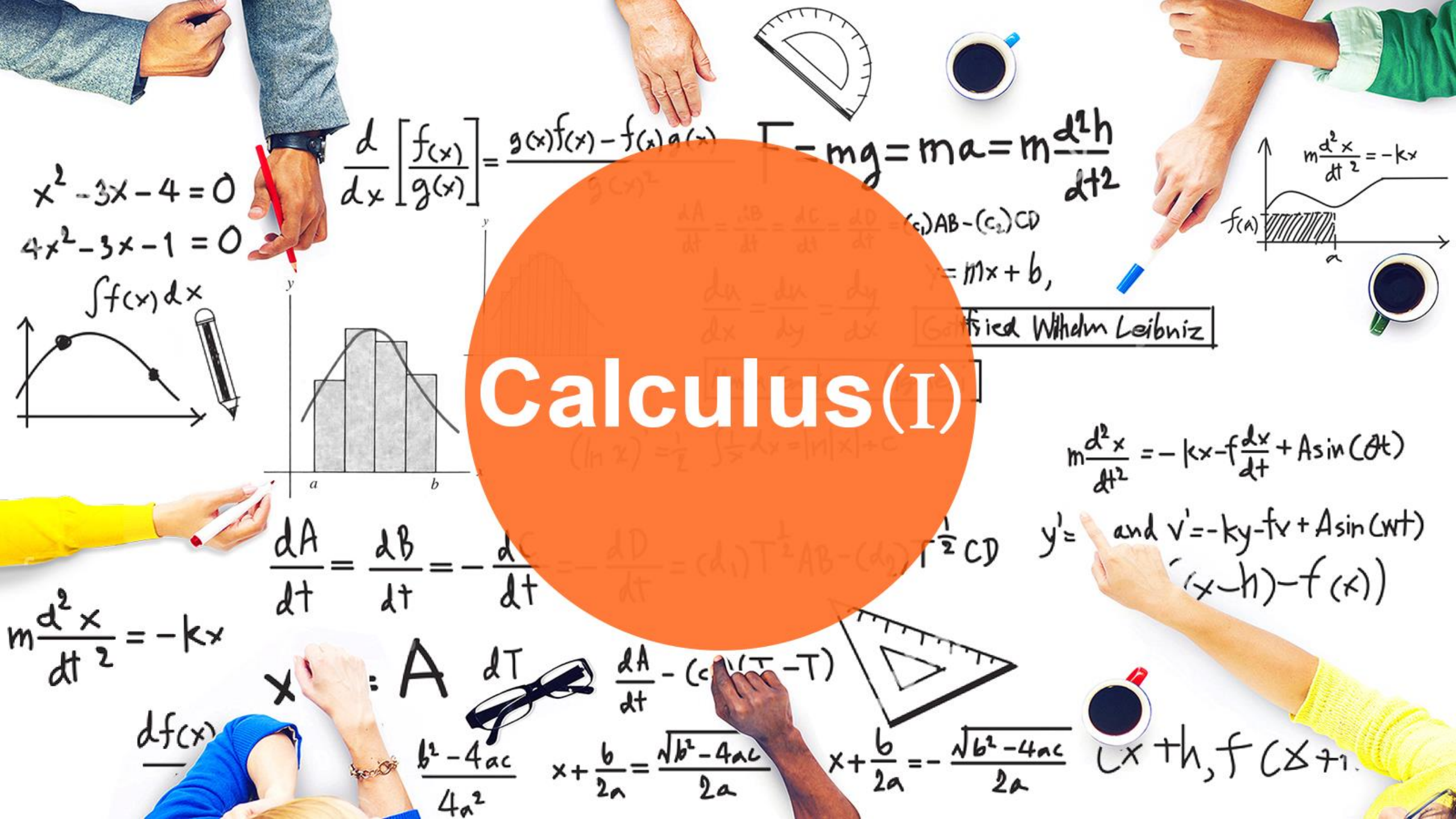

Calculus(I)

Limits Involving Trigonometric Functions

Lecturer: Xue Deng

Problem Introduction

How about $\lim_{t \to c} \sin t$, $\lim_{t \to 0} \frac{\sin t}{t}$ and $\lim_{t \to 0} \frac{1 - \cos t}{t}$?

Theorem A Limits of Trigonometric Functions

For every real number c in the function's domain

1. $\lim_{t \to c} \sin t = \sin c$
2. $\lim_{t \to c} \cos t = \cos c$
3. $\lim_{t \to c} \tan t = \tan c$
4. $\lim_{t \to c} \cot t = \cot c$
5. $\lim_{t \to c} \sec t = \sec c$
6. $\lim_{t \to c} \csc t = \csc c$

Theorem B Special Trigonometric Limits

1 $\lim_{t\to 0}\frac{\sin t}{t}=1$

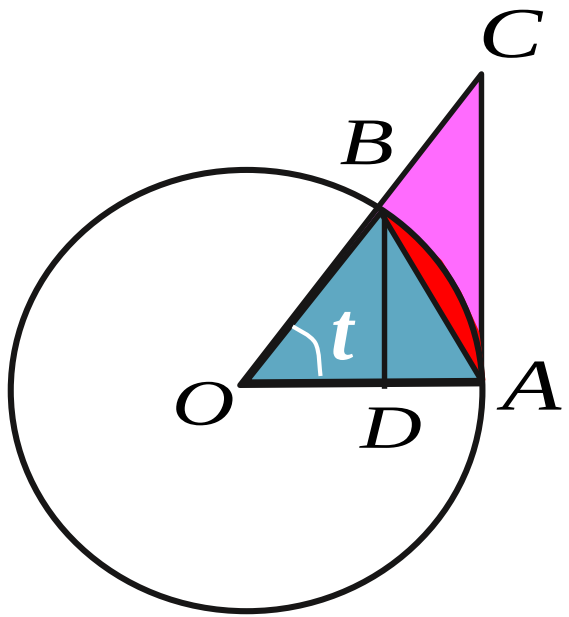
2 $\lim_{t\to 0}\frac{1-\cos t}{t}=0$

Proof: 1

Let unit-circle O, $\angle AOB = t,\ (0 < t < \frac{\pi}{2})$

Then $\sin t = BD$, $t = \overset{\frown}{AB}$, $\tan t = AC$,

Area of $\Delta AOB <$ Area of Sector $AOB <$ Area of ΔAOC



Theorem B Special Trigonometric Limits

namely $\frac{1}{2}\sin t < \frac{1}{2}t < \frac{1}{2}\tan t$

$$\therefore \sin t < t < \tan t, \quad \therefore 1 < \frac{t}{\sin t} < \frac{1}{\cos t},$$

$$\cos t < \frac{\sin t}{t} < 1,$$

when $-\frac{\pi}{2} < t < 0$ this formula still holds.

$$\because \lim_{t\to 0} \cos t = 1, \qquad \because \lim_{t\to 0} 1 = 1,$$

By Squeeze Theorem

$$\therefore \lim_{t\to 0} \frac{\sin t}{t} = 1 \quad \left(\lim_{t\to 0} \frac{t}{\sin t} = 1 \right)$$

Theorem B Special Trigonometric Limits

$$\lim_{x\to 0}\frac{\sin\square}{\square}=1$$

(1) It is a limit of type $\left[\frac{0}{0}\right]$;

(2) sin □ and □, the corresponding □ are same expressions, tend to 0

$$\lim_{x\to\infty}\frac{\sin x}{x}=1$$ ✗ ? (∵ It is not the type of $\left[\frac{0}{0}\right]$.)

$$\lim_{x\to\infty}\frac{\sin x}{x}=0$$ ✓

Example 1

Find $\lim_{t \to 0} \frac{t^2}{t+1} \cdot \cos t$

$$= \lim_{t \to 0} \frac{t^2}{t+1} \cdot \lim_{t \to 0} \cos t$$

$$= 0 \cdot 1$$

$$= \mathbf{0}.$$

Example 2

(1) $\lim\limits_{x\to 0}\dfrac{x}{\tan x}$

$$= \lim_{x\to 0}\frac{x}{\sin x}\cos x$$

$$= \lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\cos x$$

$$= 1.$$

(2) $\lim\limits_{x\to 0}\dfrac{\sin^3\sqrt[3]{x}}{3x}$

$$= \frac{1}{3}\lim_{x\to 0}\left(\frac{\sin\sqrt[3]{x}}{\sqrt[3]{x}}\right)^3$$

$$= \frac{1}{3}.$$

Example 3

(1) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}.$$

(2) $\lim\limits_{n\to\infty} n\sin\dfrac{2}{n}$

$$= \lim_{n\to\infty} 2\frac{\sin\frac{2}{n}}{\frac{2}{n}}$$

$$= 2.$$

Summary of Theorems

1. $\lim_{t \to c} \sin t = \sin c$

2. $\lim_{t \to c} \cos t = \cos c$

3. $\lim_{t \to c} \tan t = \tan c$

4. $\lim_{t \to c} \cot t = \cot c$

5. $\lim_{t \to c} \sec t = \sec c$

6. $\lim_{t \to c} \csc t = \csc c$

7. $\lim_{t \to 0} \dfrac{\sin t}{t} = 1$

8. $\lim_{t \to 0} \dfrac{1 - \cos t}{t} = 0$

Questions and Answers

Q1: Find $\lim_{x \to 0} \frac{\sin 3x}{x}$.

$$= \lim_{x \to 0} 3 \cdot \frac{\sin 3x}{3x}$$

$$= 3 \cdot 1$$

$$= 3.$$

Questions and Answers

Q2: Find $\lim_{x\to 0}\dfrac{1-\cos x}{\sin x}$.

$$= \lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{1-\cos x}{x}$$

$$= 1\cdot 0$$

$$= 0.$$

$$\lim_{t\to 0}\frac{\sin t}{t}=1,\ \lim_{t\to 0}\frac{1-\cos t}{t}=0$$

Questions and Answers

Q3: Find $\lim_{x\to 0}\dfrac{\sin 4x}{\tan x}$.

$$= \lim_{x\to 0}\frac{\sin 4x}{\sin x/\cos x}$$

$$= \lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{4x}{\sin x}\cdot\lim_{x\to 0}\cos x$$

$$= 1\cdot 4\cdot 1$$

$$= 4.$$

Limits Involving Trigonometric Functions